

Separability and the Birkhoff-Gustavson Normalization of the Perturbed Harmonic Oscillators with Homogeneous Polynomial Potentials

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ABSTRACT. In this paper, separability of the perturbed 2-dimensional isotropic harmonic oscillators with homogeneous polynomial potentials is characterized from their Birkhoff-Gustavson (BG) normalization, one of the conventional methods for *non-integrable* Hamiltonian systems.

1. Introduction

The Bertrand-Darboux (BD) theorem is a very well-known theorem established more than a century ago ([B], [D], [MW]), which characterizes separability and existence of constants of motion quadratic in momenta of simple dynamical systems on the Euclidean plane. As expected, the BD theorem has been playing a key role of various studies on *integrable systems* (see [GPS], [H], [MW], [W] and references therein).

On turning to *non-integrable systems*, the Birkhoff-Gustavson (BG) normalization is known as one of the conventional methods to them ([M]): For a given system feasible to be normalized, the BG normalization provides a good account for the phase portrait in the regular régime.

Although directed to different characteristics of dynamical systems, those well-known methods have encountered in the inverse problem of the BG normalization which is posed by the author as follows ([UCRV], [U1]): *For a given polynomial (or power series¹) Hamiltonian in the BG normal form (BGNF), identify all the possible Hamiltonians in polynomial or in power series which share the given BGNF.* In the inverse problem of the BG normalization of the perturbed isotropic harmonic oscillators (PHOs) with homogeneous polynomial potentials of *degree-3*², the condition revealed in BD theorem (BDC) has come out as follows:

THEOREM 1.1 ([U1]). *A PHO with homogeneous polynomial potential of degree-3 shares its BGNF up to degree-4 with a PHO of degree-4 if and only if the PHO of*

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¹In discussing the BG normalization, we usually think of power series in formal sense [M].

²Throughout this paper, we deal with 2-degrees of freedom systems only.

degree-3 satisfies the ‘generic’ BDC³. The PHO of degree-4 corresponding to that PHO of degree-3 also satisfies the generic BDC.

From now on, the perturbed 2-dimensional isotropic harmonic oscillator with a homogeneous polynomial potential of degree- δ will be abbreviated to as a ‘ δ -PHO’. The aim of this paper is to report briefly that the following extension of Theorem 1.1 holds true⁴:

THEOREM 1.2 (main theorem). *For any odd δ greater than or equal to 3, a δ -PHO shares its BGNF up to degree- $(2\delta - 2)$ with a $(2\delta - 2)$ -PHO if and only if the δ -PHO is separable within a rotation of Cartesian coordinates. The $(2\delta - 2)$ -PHO sharing the BGNF with that δ -PHO is also separable within the same rotation of Cartesian coordinates.*

The organization of this paper is outlined as follows. In Section 2, the separability of the δ -PHOs is studied by applying the BD theorem to them. On associating the $2 \times (\delta - 1)$ matrix of the form

$$(1.1) \quad \mathcal{M}(K^{(\delta)}) = \begin{pmatrix} v_0^{(\delta)} - v_2^{(\delta)} & v_1^{(\delta)} - v_3^{(\delta)} & \cdots & v_{\delta-2}^{(\delta)} - v_{\delta}^{(\delta)} \\ 2v_1^{(\delta)} & 2v_2^{(\delta)} & \cdots & 2v_{\delta-1}^{(\delta)} \end{pmatrix}$$

with the δ -PHO Hamiltonian defined by

$$(1.2) \quad K^{(\delta)}(q, p) = \frac{1}{2} \sum_{j=1}^2 (p_j^2 + q_j^2) + V^{(\delta)}(q) \quad \text{and} \quad V^{(\delta)}(q) = \sum_{h=0}^{\delta} v_h^{(\delta)} \binom{\delta}{h} q_1^h q_2^{\delta-h},$$

the separability of the δ -PHOs within rotations of Cartesian coordinates is shown to be equivalent to

$$(1.3) \quad \text{rank } \mathcal{M}(K^{(\delta)}) = 1.$$

Since (1.3) with $\delta = 3, 4$ provides the ‘generic’ BDC for the 3- and 4-PHOs, the separability is taken as the extension of the ‘generic’ BDC. In Section 3, the BG normalization of the δ -PHOs is studied, which provides a plausible reason to extend the relation between the degrees, 3 and 4 of the PHOs in Theorem 1.1 to δ and $2\delta - 2$ of those in Theorem 1.2. Section 4 is devoted to the proof of Theorem 1.2: The separability is shown to be sufficient for any δ -PHO to share its BGNF up to degree- $(2\delta - 2)$ with a $(2\delta - 2)$ -PHO in subsection 4.1, and is shown to be necessary in subsection 4.2.

2. The separability of the δ -PHOs

2.1. The BD theorem for the δ -PHOs. To extend Theorem 1.1 to the PHOs of general degree, we wish to understand more the meaning of the ‘generic’ BDC for the 3- and 4-PHOs³. We start with applying the BD theorem to the δ -PHOs.

THEOREM 2.1. *A δ -PHO admits a first integral quadratic in momenta if and only if it satisfies one of the followings:*

(I) For odd $\delta \geq 5$,

$$(2.1) \quad \text{rank } \mathcal{M}(K^{(\delta)}) = 1,$$

³See Eqs. (54a) and (55b) in [U1].

⁴More detailed discussion will be made in a pair of subsequent papers, [U2].

where $\mathcal{M}(K^{(\delta)})$ is the matrix associated with the δ -PHO by (1.1) and (1.2).

(II) For even $\delta \geq 4$, one of the following (2.2) and (2.3);

$$(2.2) \quad \text{rank } \mathcal{M}(K^{(\delta)}) = 1,$$

$$(2.3) \quad \begin{cases} v_{2h}^{(\delta)} = \left\{ \binom{\delta/2}{h} / \binom{\delta}{2h} \right\} v_0^{(\delta)} & \text{with } v_0^{(\delta)} \neq 0 & (h = 1, \dots, \frac{\delta}{2}) \\ v_{2h'-1}^{(\delta)} = 0 & & (h' = 1, \dots, \frac{\delta}{2}). \end{cases}$$

(III) For $\delta = 3$, one of (2.4) and (2.5);

$$(2.4) \quad \text{rank } \mathcal{M}(K^{(3)}) = 1,$$

$$(2.5) \quad \text{rank} \begin{pmatrix} 7v_1 & -v_0 + 6v_2 & -2v_1 + 5v_3 \\ -5v_0 + 2v_2 & -6v_1 + v_3 & -7v_2 \end{pmatrix} = 1.$$

PROOF. The proof is made straightforward by writing down explicitly the BDC ([MW], [U1]) in terms of $v_h^{(\delta)}$ s.

The BDC: *There exist real-valued constants, $(\alpha, \beta, \beta', \gamma, \gamma') \neq (0, 0, 0, 0, 0)$, for which the potential function $V(q)$ of a given natural dynamical system satisfies*

$$(2.6) \quad \begin{aligned} & \left(\frac{\partial^2 V}{\partial q_2^2} - \frac{\partial^2 V}{\partial q_1^2} \right) (-2\alpha q_1 q_2 - \beta' q_2 - \beta q_1 + \gamma) \\ & + 2 \frac{\partial^2 V}{\partial q_1 \partial q_2} (\alpha q_2^2 - \alpha q_1^2 + \beta q_2 - \beta' q_1 + \gamma') \\ & + \frac{\partial V}{\partial q_1} (6\alpha q_2 + 3\beta) - \frac{\partial V}{\partial q_2} (6\alpha q_1 + 3\beta') = 0. \end{aligned}$$

(i) $\delta > 3$: On substituting

$$(2.7) \quad V(q) = \frac{1}{2} \sum_{j=1}^2 (p_j^2 + q_j^2) + V^{(\delta)}(q)$$

with (1.2) into (2.6), the lhs, denoted by $\mathcal{L}^{(\delta)}$, of (2.6) is calculated to be

$$(2.8) \quad \mathcal{L}^{(\delta)} = \mathcal{L}_\delta^{(\delta)} + \mathcal{L}_{\delta-1}^{(\delta)} + \mathcal{L}_{\delta-2}^{(\delta)} + \mathcal{L}_1^{(\delta)}$$

with the homogeneous polynomial parts,

$$(2.9) \quad \mathcal{L}_1^{(\delta)} = 3\beta q_1 - 3\beta' q_2,$$

$$(2.10) \quad \begin{aligned} \mathcal{L}_{\delta-2}^{(\delta)} &= \delta(\delta-1) \sum_{h=0}^{\delta-2} \binom{\delta-2}{h} \{ (v_h^{(\delta)} - v_{h+2}^{(\delta)}) \gamma + 2v_{h+1}^{(\delta)} \gamma' \} q_1^h q_2^{\delta-2-h} \\ &= \delta(\delta-1) \times (\gamma, \gamma') \mathcal{M}(K^{(\delta)}) (q_2^{\delta-2}, \dots, \binom{\delta-2}{h} q_1^h q_2^{\delta-2-h}, \dots, q_1^{\delta-2})^T, \end{aligned}$$

$$\begin{aligned}
(2.11) \quad \mathcal{L}_{\delta-1}^{(\delta)} &= \beta\delta \left[(2\delta+1)v_1^{(\delta)}q_2^{\delta-1} + \left\{ (\delta+2)v_\delta^{(\delta)} - (\delta-1)v_{\delta-2}^{(\delta)} \right\} q_1^{\delta-1} \right. \\
&\quad + \sum_{h=1}^{\delta-2} \left\{ \left(3\delta \binom{\delta-1}{h} - (\delta-1) \binom{\delta-2}{h} \right) v_{h+1}^{(\delta)} - 3(\delta-1) \binom{\delta-2}{h-1} v_{h-1}^{(\delta)} \right\} q_1^h q_2^{\delta-1-h} \Big] \\
&\quad - \beta'\delta \left[\left\{ (\delta+2)v_0^{(\delta)} - (\delta-1)v_2^{(\delta)} \right\} q_2^{\delta-1} + (2\delta+1)v_1^{(\delta)}q_1^{\delta-1} \right. \\
&\quad \left. - \sum_{h=1}^{\delta-2} \left\{ \left(3\delta \binom{\delta-1}{h} - (\delta-1) \binom{\delta-2}{h-1} \right) v_h^{(\delta)} - 3(\delta-1) \binom{\delta-2}{h} v_{h+2}^{(\delta)} \right\} q_1^h q_2^{\delta-1-h} \right],
\end{aligned}$$

and

$$\begin{aligned}
(2.12) \quad \mathcal{L}_\delta^{(\delta)} &= 2\alpha\delta(\delta+2) \\
&\quad \times \left[v_1^{(\delta)}q_2^\delta - v_{\delta-1}^{(\delta)}q_1^\delta + \sum_{h=1}^{\delta-1} \left\{ \binom{\delta-1}{h} v_{h+1}^{(\delta)} - \binom{\delta-1}{h-1} v_{h-1}^{(\delta)} \right\} q_1^h q_2^{\delta-h} \right],
\end{aligned}$$

of degree-1, $-(\delta-2)$, $-(\delta-1)$ and $-\delta$, respectively ⁵.

(ii) $\delta = 3$: Substituting (2.7) with $\delta = 3$ into (2.6), we obtain

$$(2.13) \quad \mathcal{L}^{(3)} = \mathcal{L}_3^{(3)} + \mathcal{L}_2^{(3)} + \mathcal{L}_1^{(3)}$$

with

$$(2.14) \quad \mathcal{L}_1^{(3)} = 3 \left\{ (\gamma, \gamma') \mathcal{M}(K^{(3)}) + (-\beta', \beta) \right\} \begin{pmatrix} q_2 \\ q_1 \end{pmatrix}$$

and

$$(2.15) \quad \mathcal{L}_2^{(3)} = 3(\beta, \beta') \begin{pmatrix} 7v_1 & -v_0 + 6v_2 & -2v_1 + 5v_3 \\ -5v_0 + 2v_2 & -6v_1 + v_3 & -7v_2 \end{pmatrix} (q_2^2, 2q_1q_2, q_1^2)^T,$$

where $\mathcal{L}_3^{(3)}$ is given by (2.12) with $\delta = 3$.

From the explicit expression of $\mathcal{L}^{(\delta)}$ thus obtained, we have Table 1, which classifies the possible choice of $(\alpha, \beta, \beta', \gamma, \gamma') \neq 0$.

TABLE 1. The BDC for the δ -PHOs.

	δ :odd (≥ 5)	δ :even (≥ 4)	$\delta = 3$
$\alpha = 0, (\beta, \beta') = (0, 0), (\gamma, \gamma') \neq (0, 0)$	(2.1)	(2.2)	(2.4)
$\alpha = 0, (\beta, \beta') \neq (0, 0), (\gamma, \gamma') \neq (0, 0)$	—	—	(2.5)
$\alpha \neq 0, (\beta, \beta') = (0, 0), (\gamma, \gamma') = (0, 0)$	—	(2.3)	—

The derivation of Table 2.1 will be given in more detail in [U2]. □

2.2. The separability. As expected from the BD theorem, the classification, (2.1)-(2.5), of the BDC can be characterized from the separability viewpoint.

THEOREM 2.2. *A δ -PHO is separable ⁶ within a rotation of Cartesian coordinates if and only if (1.3) holds true.*

⁵The superscript T stands for the transpose throughout this paper.

⁶As known well, a Hamiltonian system is said to be separable iff the Hamilton-Jacobi equation for that system is separable.

PROOF. From the Hamilton-Jacobi equation

$$(2.16) \quad \frac{1}{2} \sum_{j=1}^2 \left(\frac{\partial S}{\partial q_j} \right)^2 + V^{(\delta)}(q) = E \quad (E : \text{energy value})$$

for the δ -PHOs ($S(q)$: the generating function), it is easy to see that the separation of (2.16) with in rotations of Cartesian coordinates amounts to that of $V^{(\delta)}(q)$. Accordingly, let us assume that $V^{(\delta)}(q)$ is separated within a rotation

$$(2.17) \quad \kappa_\psi : (q, p) \rightarrow (\tilde{q}, \tilde{p}) = (\sigma(\psi)q, \sigma(\psi)p)$$

with

$$(2.18) \quad \sigma(\psi) = \begin{pmatrix} \cos \psi & -\sin \psi \\ \sin \psi & \cos \psi \end{pmatrix} \quad (0 \leq \psi < 2\pi).$$

Namely,

$$(2.19) \quad \tilde{V}^{(\delta)}(\tilde{q}) = V^{(\delta)}(\sigma(\psi)^{-1}\tilde{q}) = \tilde{v}_0^{(\delta)}\tilde{q}_2^\delta + \tilde{v}_\delta^{(\delta)}\tilde{q}_1^\delta,$$

where $(\tilde{v}_0^{(\delta)}, \tilde{v}_\delta^{(\delta)}) \neq (0, 0)$. Equations (2.17)-(2.19) are put together to yield the relation,

$$(2.20) \quad (4 \sin 2\psi)(v_h^{(\delta)} - v_{h+2}^{(\delta)}) - (\cos 2\psi)(2v_{h+1}^{(\delta)}) = 0 \quad (h = 0, \dots, \delta - 2)$$

among $v_h^{(\delta)}$ s, which immediately implies (1.3). The converse is easily shown by tracing back the discussion above. \square

REMARK 2.3. It is also possible to characterize the other classes of δ -PHOs subject to BDC listed in Theorem 2.6 from the separation of variables viewpoint: The condition (2.3) is shown to be equivalent to the separability in the polar coordinates, and (2.5) in a off-centered parabolic coordinates ([U2]).

Since the ‘generic’ BDC for the 3-PHOs and the 4-PHOs³ are equivalent to (1.3) with $\delta = 3, 4$, we can thereby look the separability within rotations upon as the ‘generic’ BDC for δ -PHOs owing to Theorem 2.2.

3. The BG normalization of the δ -PHO

In this section, we proceed the BG normalization of the δ -PHOs, which provides us with a key other than Theorem 2.2 to extend Theorem 1.1 to Theorem 1.2.

We start with describe the way how the δ -PHO Hamiltonian $K^{(\delta)}(q, p)$ is brought into the BGNF. Let $G^{(\delta)}(\xi, \eta)$ be the BGNF of $K^{(\delta)}(q, p)$ and $W^{(\delta)}(q, \eta)$ be the generating function⁷ used for the BG normalization, both of which are expressed in power-series form¹

$$(3.1) \quad G^{(\delta)}(\xi, \eta) = \frac{1}{2} \sum_{j=1}^2 (\eta_j^2 + \xi_j^2) + \sum_{k=3}^{\infty} G_k^{(\delta)}(\xi, \eta)$$

and

$$(3.2) \quad W^{(\delta)}(q, \eta) = \sum_{j=1}^2 q_j \eta_j + \sum_{k=3}^{\infty} W_k^{(\delta)}(q, \eta),$$

⁷ $W^{(\delta)}(q, \eta)$ is said to be of the second-type since it is a function of the ‘old’ position variables q and the ‘new’ momentum ones η ([G]).

where $G_k^{(\delta)}(\xi, \eta)$ and $W_k^{(\delta)}(q, \eta)$ denote the homogeneous polynomial parts of degree- k of $G^{(\delta)}(\xi, \eta)$ and $W^{(\delta)}(q, \eta)$, respectively. We normalize $K^{(\delta)}(q, p)$ by applying the canonical transformation,

$$(3.3) \quad \tau : (q, p) \rightarrow (\xi, \eta) \quad \text{with} \quad p = \frac{\partial W^{(\delta)}}{\partial q} \quad \text{and} \quad \xi = \frac{\partial W^{(\delta)}}{\partial \eta}$$

associated with $W^{(\delta)}(q, \eta)$. Namely, $G^{(\delta)}(\xi, \eta)$ is determined by

$$(3.4) \quad G^{(\delta)}\left(\frac{\partial W^{(\delta)}}{\partial \eta}, \eta\right) = K^{(\delta)}\left(q, \frac{\partial W^{(\delta)}}{\partial q}\right).$$

DEFINITION 3.1 (The BGNF). Let $G^{(\delta)}(\xi, \eta)$ be the power-series¹ (3.1), where each $G_k^{(\delta)}(\xi, \eta)$ is a homogeneous polynomial of degree- k ($k = 3, 4, \dots$) in (ξ, η) ⁸. Then $G^{(\delta)}(\xi, \eta)$ is said to be in the BGNF up to degree- ρ if and only if it satisfies

$$(3.5) \quad G_k^{(\delta)}(\xi, \eta) \in \ker D_{\xi, \eta}^{(k)} \quad (k = 3, \dots, \rho),$$

where $D_{\xi, \eta}^{(k)}$ is the restrict of the linear differential operator

$$(3.6) \quad D_{\xi, \eta} = \sum_{j=1}^2 \left(\xi_j \frac{\partial}{\partial \eta_j} - \eta_j \frac{\partial}{\partial \xi_j} \right)$$

on the vector space of homogeneous polynomials of degree- k in (ξ, η) .

REMARK 3.2. The $D_{\xi, \eta}$ is understood as the Poisson derivation $([\mathbf{A}])$, $D_{\xi, \eta} = \{\frac{1}{2} \sum_{j=1}^2 (\eta_j^2 + \xi_j^2), \cdot\}$, associated with the isotropic harmonic oscillator.

The ordinary problem of the BG normalization of the δ -PHOs is posed as follows⁹:

DEFINITION 3.3 (The ordinary problem of degree- ρ , $[\mathbf{UCRV}]$, $[\mathbf{U1}]$). Bring a given δ -PHO Hamiltonian $K^{(\delta)}(q, p)$ of the form (1.2) into the power series, $G^{(\delta)}(\xi, \eta)$, in the BGNF up to degree- ρ through (3.4), where generating function $W^{(\delta)}(q, \eta)$ of the second-type in the form (3.2) is chosen to satisfy (3.4) and

$$(3.7) \quad W_k^{(\delta)}(q, \eta) \in \text{image} D_{q, \eta}^{(k)} \quad (k = 3, 4, \dots, \rho).$$

The $D_{q, \eta}^{(k)}$ is defined by (3.6) with q in place of ξ .

REMARK 3.4. The condition (3.7) is very crucial to ensure the uniqueness of the outcome, $G^{(\delta)}(\xi, \eta)$, from $K^{(\delta)}(q, p)$ ($[\mathbf{UCRV}]$, $[\mathbf{U1}]$).

We are now in a position to present an explicit expression of the BGNF $G^{(\delta)}(\xi, \eta)$ of the δ -PHO Hamiltonian. A straightforward calculation of (3.4) shows the following:

LEMMA 3.5. *The BGNF, $G^{(\delta)}(\xi, \eta)$, of the δ -PHO Hamiltonian $K^{(\delta)}(q, p)$ in the form (1.2) takes the form*

$$(3.8) \quad G^{(\delta)}(\xi, \eta) = \frac{1}{2} \sum_{j=1}^2 (\eta_j^2 + \xi_j^2) + G_\delta^{(\delta)}(\xi, \eta) + G_{2\delta-2}^{(\delta)}(\xi, \eta) + o_{2\delta-1}(\xi, \eta),$$

⁸The homogeneous part of degree-2 in $G^{(\delta)}(\xi, \eta)$ is always in the isotropic harmonic oscillator form, due to (3.2).

⁹For the inverse problem of the BG normalization, see $[\mathbf{UCRV}]$ and $[\mathbf{U2}]$.

where $o_{2\delta-1}(\xi, \eta)$ denotes a power series in (ξ, η) starting from degree- $(2\delta-1)$. The homogeneous polynomial part, $G_\delta^{(\delta)}(\xi, \eta)$, of degree- δ is given by

$$(3.9) \quad G_\delta^{(\delta)}(\xi, \eta) = V^{(\delta)\ker}(\xi, \eta) = \begin{cases} 0 & (\delta: \text{ odd}) \\ 2^{-\delta} \sum_{h=0}^{\delta} \binom{\delta}{h} v_h^{(\delta)} \left[\sum_{m=M^\sharp}^{M^\sharp} \sum_{n=N^\sharp}^{N^\sharp} \binom{h}{m} \binom{\delta-h}{n} \zeta_1^m \zeta_2^n \bar{\zeta}_1^{h-m} \bar{\zeta}_2^{\delta-h-n} \right] & (\delta: \text{ even}), \end{cases}$$

where $\zeta_j = \xi_j + i\eta_j$ ($j = 1, 2$), and the ranges of the summation indices, m and n , are determined by

$$(3.10) \quad \begin{cases} M^\sharp = \max(0, h - \frac{\delta}{2}) & M^\sharp = \min(h, \frac{\delta}{2}) \\ N^\sharp = \max(0, \frac{\delta}{2} - h) & N^\sharp = \min(\delta - h, \frac{\delta}{2}). \end{cases}$$

The superscript \ker stands for taking the kernel component of $V^{(\delta)}(q)$ according to the action of $D_{q,\eta}^{(\delta)}$. The homogeneous polynomial part, $G_{2\delta-2}^{(\delta)}(\xi, \eta)$, of degree- $(2\delta-2)$ is calculated to be

$$(3.11) \quad G_{2\delta-2}^{(\delta)}(\xi, \eta) = \sum_{m=0}^{2\delta-2} \sum_{\ell=L^\sharp}^{L^\sharp} c_{m,\ell}^{(\delta)} \zeta_1^\ell \zeta_2^{\delta-1-\ell} \bar{\zeta}_1^{m-\ell} \bar{\zeta}_2^{\delta-1-m+\ell}$$

with

$$(3.12) \quad c_{m,\ell}^{(\delta)} = \frac{2\delta^2}{4\delta} \sum_{j=J^\sharp}^{J^\sharp} \left[\binom{\delta-1}{j} \binom{\delta-1}{m-j} (v_j^{(\delta)} v_{m-j}^{(\delta)} + v_{j+1}^{(\delta)} v_{m-j+1}^{(\delta)}) \right. \\ \left. \times \sum_{k=K^\sharp}^{K^\sharp} \sum_{h=H^\sharp}^{H^\sharp} \frac{\binom{j}{k} \binom{\delta-1-j}{h} \binom{m-j}{\ell-k} \binom{\delta-1-(m-j)}{\delta-1-\ell-h}}{\{(2(k+h+1)-\delta)\}\{2(k+h)-\delta\}} \right].$$

The ranges of the indices, h, j, k and ℓ , in (3.11) and (3.12) are determined by

$$(3.13) \quad \begin{cases} H^\sharp = \max(0, (m-j) - \ell) & H^\sharp = \min(\delta-1-\ell, \delta-1-j) \\ J^\sharp = \max(0, m - (\delta-1)) & J^\sharp = \min(m, \delta-1) \\ K^\sharp = \max(0, \ell - (m-j)) & K^\sharp = \min(j, \ell) \\ L^\sharp = \max(0, m - (\delta-1)), & L^\sharp = \min(m, \delta-1) \end{cases}$$

respectively.

The proof is outlined in Appendix A and will be given in more detail in [U2]. From Lemma 3.5, we obtain the following:

THEOREM 3.6. *Let δ_1 and δ_2 be any integers subject to*

$$(3.14) \quad 3 \leq \delta_1 < \delta_2.$$

If the BGNF, $G^{(\delta_1)}(\xi, \eta)$, of a δ_1 -PHO Hamiltonian $K^{\delta_1}(q, p)$ coincides with the BGNF, $G^{(\delta_2)}(\xi, \eta)$, of a δ_2 -PHO Hamiltonian $K^{(\delta_2)}(q, p)$ up to degree- δ_2 , then δ_1 and δ_2 have to satisfy

$$(3.15) \quad \delta_1 : \text{ an odd integer } \quad \text{and} \quad \delta_2 = 2\delta_1 - 2.$$

PROOF. Since the coincidence of the BGNFs is expressed as

$$(3.16) \quad G^{(\delta_1)}(\xi, \eta) - G^{(\delta_2)}(\xi, \eta) = o_{\delta_2+1}(\xi, \eta),$$

we obtain

$$(3.17) \quad G_{\delta_1}^{(\delta_1)}(\xi, \eta) = G_{\delta_1}^{(\delta_2)}(\xi, \eta) = 0$$

from Lemma 3.5 as a necessary condition for (3.16). Equation (3.17) is put together with (3.9) to yield the first condition in (3.15). We derive the second one in turn. On recalling Lemma 3.5 again, the lowest non-vanishing homogeneous polynomial part of $G^{(\delta_1)}(\xi, \eta)$ turns out to be $G_{2\delta_1-2}^{(\delta_1)}(\xi, \eta)$ if δ_1 is odd, while $G_{\delta_2}^{(\delta_2)}(\xi, \eta)$ is the lowest one of $G^{(\delta_2)}(\xi, \eta)$. This shows the second equation of (3.15). \square

Now that we have a pair of key Theorems 2.2 and 3.6, we are led to pose Theorem 1.2 as an extension of Theorem 1.1.

4. Proof of the main theorem

In this section, the proof of Theorem 1.2 is outlined. Throughout this section, we assume $\delta \geq 3$ to be odd.

4.1. Part I: the separability as a sufficiency. This subsection is devoted to show that the separability is sufficient for a δ -PHO to share its BGNF with a $(2\delta - 2)$ -PHO up to degree- $(2\delta - 2)$.

REMARK 4.1. Recalling the proof of Theorem 2.2, we see that the separability of the δ -PHOs⁵ within rotations of Cartesian coordinates is equivalent to the separability of their Hamiltonians. We will use those equivalent expressions properly according to circumstances henceforce.

4.1.1. *The BG normalization of the δ -PHOs in separate form.* Let the δ -PHO be associated with the Hamiltonian in separate form,

$$(4.1) \quad K_{\text{sep}}^{(\delta)}(q, p) = \frac{1}{2} \sum_{j=1}^2 (p_j^2 + q_j^2) + (v_{\text{sep},0}^{(\delta)} q_2^\delta + v_{\text{sep},\delta}^{(\delta)} q_1^\delta).$$

Then, applying (3.11) and (3.12) to $K_{\text{sep}}^{(\delta)}(q, p)$, we obtain the BGNF of $K_{\text{sep}}^{(\delta)}(q, p)$, denoted by $G_{\text{sep}}^{(\delta)}(\xi, \eta)$, to be in the following separate form,

$$(4.2) \quad G_{\text{sep}}^{(\delta)}(\xi, \eta) = \frac{1}{2} \sum_{j=1}^2 (\eta_j^2 + \xi_j^2) + \left\{ \frac{2\delta^2}{4^\delta} \sum_{n=0}^{\delta-1} \frac{\binom{\delta-1}{n}^2}{\{2(n+1)-\delta\}(2n-\delta)} \right\} \\ \times \{v_{\text{sep},0}^{(\delta)} {}^2(\zeta_2 \bar{\zeta}_2)^{\delta-1} + v_{\text{sep},\delta}^{(\delta)} {}^2(\zeta_1 \bar{\zeta}_1)^{\delta-1}\} + o_{2\delta-1}(\xi, \eta),$$

where $\zeta_j = \xi + \eta_j$ ($j = 1, 2$). Recalling (3.8) and (3.9) with $2\delta - 2$ in place of δ , we can find the unique $(2\delta - 2)$ -PHO Hamiltonian

$$(4.3) \quad K_{\text{sep}}^{(2\delta-2)}(q, p) = \frac{1}{2} \sum_{j=1}^2 (p_j^2 + q_j^2) + \left\{ \frac{\delta^2}{2\binom{2\delta-2}{\delta-1}} \sum_{n=0}^{\delta-1} \frac{\binom{\delta-1}{n}^2}{\{2(n+1)-\delta\}(2n-\delta)} \right\} \\ \times \{(v_{\text{sep},0}^{(\delta)})^2 q_2^{2\delta-2} + (v_{\text{sep},\delta}^{(\delta)})^2 q_1^{2\delta-2}\}$$

in separate form, whose BGNF coincides with $G_{\text{sep}}^{(\delta)}(\xi, \eta)$ up to degree- $(2\delta - 2)$. To summarize, we have the following.

LEMMA 4.2. For any δ -PHO Hamiltonian $K_{\text{sep}}^{(\delta)}(q, p)$ in separate form (4.1), there exists the unique $(2\delta - 2)$ -PHO Hamiltonian $K_{\text{sep}}^{(2\delta-2)}(q, p)$ in separate form (4.3) which shares the BGNF $G_{\text{sep}}^{(\delta)}(\xi, \eta)$ up to degree- $(2\delta - 2)$ with $K_{\text{sep}}^{(\delta)}(q, p)$.

4.1.2. *Proof of the sufficiency.* A key to prove the sufficiency is the commutativity of the BG normalization and the rotations of Cartesian coordinates:

LEMMA 4.3. Let $G^{(\delta)}(\xi, \eta)$ be the BGNF of a δ -PHO Hamiltonian $K^{(\delta)}(q, p)$ up to degree- $(2\delta - 2)$, which associates with the generating function $W^{(\delta)}(q, \eta)$ (see (3.4)). Let $\tilde{G}^{(\delta)}(\tilde{\xi}, \tilde{\eta})$, $\tilde{K}^{(\delta)}(\tilde{q}, \tilde{p})$ and $\tilde{W}^{(\delta)}(\tilde{q}, \tilde{\eta})$ be the power series defined by

$$(4.4) \quad \begin{aligned} \tilde{K}^{(\delta)}(\tilde{q}, \tilde{p}) &= K^{(\delta)}(\sigma^{-1}(\psi)\tilde{q}, \sigma^{-1}(\psi)\tilde{p}) \\ \tilde{G}^{(\delta)}(\tilde{\xi}, \tilde{\eta}) &= G^{(\delta)}(\sigma^{-1}(\psi)\tilde{\xi}, \sigma^{-1}(\psi)\tilde{\eta}) \\ \tilde{W}^{(\delta)}(\tilde{q}, \tilde{\eta}) &= W^{(\delta)}(\sigma^{-1}(\psi)\tilde{q}, \sigma^{-1}(\psi)\tilde{\eta}), \end{aligned}$$

where $\sigma(\psi)$ is defined by (2.18). Then $\tilde{G}^{(\delta)}(\tilde{\xi}, \tilde{\eta})$ is the BGNF of the δ -PHO Hamiltonian $\tilde{K}^{(\delta)}(\tilde{q}, \tilde{p})$ up to degree- $(2\delta - 2)$, which is brought through the canonical transformation, $(\tilde{q}, \tilde{p}) \rightarrow (\tilde{\xi}, \tilde{\eta})$, generated by $\tilde{W}^{(\delta)}(\tilde{q}, \tilde{\eta})$.

PROOF. Due to the orthogonality of $\sigma(\psi)$, it is easily confirmed from (4.4) that $\tilde{K}^{(\delta)}(\tilde{q}, \tilde{p})$, $\tilde{G}^{(\delta)}(\tilde{\xi}, \tilde{\eta})$ and $\tilde{W}^{(\delta)}(\tilde{q}, \tilde{\eta})$ are other δ -PHO Hamiltonian, BGNF up to degree- $(2\delta - 2)$ and generating function of the second-type (cf. (1.2), (3.1) and (3.2)), respectively. Further, the orthogonality of $\sigma(\psi)$ is put together with (4.4) to yield the equation,

$$(4.5) \quad \tilde{K}^{(\delta)}(\tilde{q}, \frac{\partial \tilde{W}^{(\delta)}}{\partial \tilde{q}}) = \tilde{G}^{(\delta)}(\frac{\partial \tilde{W}^{(\delta)}}{\partial \tilde{\eta}}, \tilde{\eta}),$$

from (3.4), so that $\tilde{G}^{(\delta)}(\tilde{\xi}, \tilde{\eta})$ is the BGNF of $\tilde{K}^{(\delta)}(\tilde{q}, \tilde{p})$. This completes the proof. \square

We are at the final stage to prove the sufficiency of the separability in Theorem 1.2 now. Let us assume that $K^{(\delta)}(q, p)$ is separable within a rotation of Cartesian coordinates: Namely, there exists the transformation κ_ψ with a suitable $\psi \in [0, 2\pi)$ (see (2.17) and (2.18)) which brings $K^{(\delta)}(q, p)$ to $K_{\text{sep}}^{(\delta)}(\tilde{q}, \tilde{p})$ through

$$(4.6) \quad K_{\text{sep}}^{(\delta)}(\tilde{q}, \tilde{p}) = K^{(\delta)}(\sigma(\psi)^{-1}\tilde{q}, \sigma(\psi)^{-1}\tilde{p}),$$

where $K_{\text{sep}}^{(\delta)}(\tilde{q}, \tilde{p})$ takes the separate form (4.1) with (\tilde{q}, \tilde{p}) in place of (q, p) . Then on applying Lemma 4.3 to the pair, $K_{\text{sep}}^{(\delta)}(\tilde{q}, \tilde{p})$ and $G_{\text{sep}}^{(\delta)}(\tilde{\xi}, \tilde{\eta})$, the BGNF $G^{(\delta)}(\xi, \eta)$ of $K^{(\delta)}(q, p)$ up to degree- $(2\delta - 2)$ is given by

$$(4.7) \quad G^{(\delta)}(\xi, \eta) = G_{\text{sep}}^{(\delta)}(\sigma(\psi)\xi, \sigma(\psi)\eta).$$

Further, according to Lemma 4.2, we can find uniquely the $(2\delta - 2)$ -PHO Hamiltonian in separate form, say $K_{\text{sep}}^{(2\delta-2)}(\tilde{q}, \tilde{p})$, sharing the BGNF $G_{\text{sep}}^{(\delta)}(\tilde{\xi}, \tilde{\eta})$ up to degree- $(2\delta - 2)$ with $K_{\text{sep}}^{(\delta)}(\tilde{q}, \tilde{p})$. On defining the $(2\delta - 2)$ -PHO Hamiltonian $K^{(2\delta-2)}(q, p)$ and the BGNF $G^{(2\delta-2)}(\xi, \eta)$ by

$$(4.8) \quad \begin{aligned} K^{(2\delta-2)}(q, p) &= K_{\text{sep}}^{(2\delta-2)}(\sigma(\psi)q, \sigma(\psi)p) \\ G^{(2\delta-2)}(\xi, \eta) &= G_{\text{sep}}^{(2\delta-2)}(\sigma(\psi)\xi, \sigma(\psi)\eta), \end{aligned}$$

Lemma 4.3 shows that $G^{(2\delta-2)}(\xi, \eta)$ is the BGNF of $K^{(2\delta-2)}(q, p)$ up to degree- $(2\delta - 2)$. Equations (4.7) and (4.8) are put together to show the coincidence of

$G^{(2\delta-2)}(\xi, \eta)$ with $G^{(\delta)}(\xi, \eta)$ up to degree- $(2\delta-2)$. To summarize, we have the following.

THEOREM 4.4. *Let δ be an odd integer greater than or equal to 3. If a δ -PHO Hamiltonian is separable (see Remark 4.1) within a rotation of Cartesian coordinates, there exists the unique $(2\delta-2)$ -PHO Hamiltonian which shares the same BGNF up to degree- $(2\delta-2)$ with the δ -PHO. The $(2\delta-2)$ -PHO is also separable within the same rotation of Cartesian coordinates.*

4.2. Part II: the separability as a necessity. This subsection is devoted to prove that the separability is a necessary condition. Let us recall Lemma 3.5 and equate $G^{(\delta)}(\xi, \eta)$ with $G^{(2\delta-2)}(\xi, \eta)$ up to degree- $(2\delta-2)$. As a necessary and sufficient condition for the equation thus obtained, we have a number of equations

$$(4.9) \quad c_{m,\ell}^{(\delta)} = c_{m,\ell}^{(2\delta-2)} \quad (0 \leq m \leq 2\delta-2, L^b \leq \ell \leq L^\sharp).$$

Since the number of equations in (4.9) is so many and since we have already shown the separability as a sufficiency, it would not be so smart to study (4.9) for all the pairs of subscripts, (m, ℓ) s. Hence as necessary condition for (4.9), we consider

$$(4.10) \quad c_{m,1}^{(\delta)} \tilde{c}_{m,0}^{(2\delta-2)} = c_{m,0}^{(\delta)} \tilde{c}_{m,1}^{(2\delta-2)} \quad (m = 2, 3, \dots, \delta-1)$$

with

$$(4.11) \quad \tilde{c}_{m,\ell}^{(2\delta-2)} = 4^{1-\delta} \binom{2\delta-2}{m} \binom{m}{\delta-1-\ell} \quad (\ell = 0, 1).$$

Note that we have $c_{m,\ell}^{(2\delta-2)} = v_m^{(2\delta-2)} \tilde{c}_{m,\ell}^{(2\delta-2)}$ for $\ell = 0, 1$.

We wish to draw (1.3) as a necessary condition from (4.10). To do this, it is useful to prepare the notation,

$$(4.12) \quad U_j^{(\delta,m)} = (v_j^{(\delta)} v_{m-j}^{(\delta)} + v_{j+1}^{(\delta)} v_{m-j+1}^{(\delta)} - (v_{j+1}^{(\delta)} v_{m-(j+1)}^{(\delta)} + v_{(j+1)+1}^{(\delta)} v_{m-(j+1)+1}^{(\delta)}) \quad (j = 0, 1, \dots, m-1).$$

Using (3.9), (3.11) and (4.12), we can put (4.10) into the form

$$(4.13) \quad \left(\sum_{n=0}^m B_n^{(\delta,m)} \right) (v_m^{(\delta)} v_0^{(\delta)} + v_{m+1}^{(\delta)} v_1^{(\delta)}) + \sum_{j=0}^{m-1} \left(\sum_{n=0}^j B_n^{(\delta,m)} \right) U_j^{(\delta,m)} = 0 \quad (m = 2, \dots, \delta-1),$$

where $B_n^{(\delta, m)}$ s are defined to be

$$(4.14) \quad \left((\delta - 1)^2 \binom{\delta-2}{m-1} \right)^{-1} B_n^{(\delta, m)} = \begin{cases} \sum_{k=m-n-1}^{\delta-n-1} \frac{\binom{m-1}{n} \binom{\delta-m}{n+k+1-m} \binom{\delta-2}{k}}{\{2(k+1)-\delta\}(2k-\delta)} \\ + \sum_{k=m-n-1}^{\delta-n-1} \frac{\binom{m-1}{n-1} \binom{\delta-m}{n+k+1-m} \binom{\delta-2}{k}}{\{2(k+2)-\delta\}\{2(k+1)-\delta\}} \\ - \sum_{k=m-n}^{\delta-h-1} \frac{\binom{m}{n} \binom{\delta-m-1}{n+k-m} \binom{\delta-1}{k}}{\{2(k+1)-\delta\}(2k-\delta)} \quad (n \neq 0, m), \\ \\ \sum_{k=m-1}^{\delta-2} \frac{\binom{\delta-m}{k+1-m} \binom{\delta-2}{k}}{\{2(k+1)-\delta\}(2k-\delta)} \\ - \sum_{k=m}^{\delta-1} \frac{\binom{\delta-m-1}{k-m} \binom{\delta-1}{k}}{\{2(k+1)-\delta\}(2k-\delta)} \quad (n = 0), \\ \\ \sum_{k=0}^{\delta-m-1} \frac{\binom{\delta-m}{k+1} \binom{\delta-2}{k}}{\{2(k+2)-\delta\}\{2(k+1)-\delta\}} \\ - \sum_{k=0}^{\delta-m-1} \frac{\binom{\delta-m-1}{k+1} \binom{\delta-1}{k}}{\{2(k+1)-\delta\}(2k-\delta)} \quad (n = m). \end{cases}$$

The definition (4.14) of $B_n^{(\delta, m)}$ s and several well-known formula for the binomial coefficients are put together to show

$$(4.15) \quad \sum_{n=0}^m B_n^{(\delta, m)} = 0 \quad (m = 2, \dots, \delta - 1)$$

(see Appendix B), so that we can put (4.13) in the form

$$(4.16) \quad \sum_{j=0}^{m-1} \left(\sum_{n=0}^j B_n^{(\delta, m)} \right) U_j^{(\delta, m)} = 0 \quad (m = 2, \dots, \delta - 1).$$

We verify (4.16) further by characterizing $U_j^{(\delta, m)}$ as the minors of $\mathcal{M}(K^{(\delta)})$ as follows. Denoting by $\Delta_{ab}^{(\delta)}$ the minor of $\mathcal{M}(K^{(\delta)})$ consisting of its a - and b -th columns ($1 \leq a < b \leq \delta - 1$), we have

$$(4.17) \quad U_j^{(\delta, m)} = \begin{cases} \Delta_{j+1, m-j}^{(\delta)} & (j = 0, \dots, [m/2] - 1) \\ -\Delta_{m-j, j+1}^{(\delta)} & (j = [m/2], \dots, m - 1), \end{cases}$$

where $[m/2]$ stands for the integer part of $m/2$. Putting (4.17) and the symmetry,

$$(4.18) \quad \begin{aligned} U_j^{(\delta, m)} &= U_{m-j}^{(\delta, m)} \quad (j = 0, 1, \dots, m - 1) \\ B_n^{(\delta, m)} &= B_{m-n}^{(\delta, m)} \quad (n = 0, \dots, m), \end{aligned}$$

together, we can rewrite (4.16) to as

$$(4.19) \quad \sum_{j=0}^{\lfloor m/2 \rfloor - 1} \left(\sum_{n=j+1}^{m-(j+1)} B_n^{(\delta, m)} \right) \Delta_{j+1, m-j}^{(\delta)} = 0 \quad (m = 2, \dots, \delta - 1).$$

From now on, we assume

$$(4.20) \quad (v_0^{(\delta)} - v_2^{(\delta)})^2 + (v_1^{(\delta)})^2 \neq 0$$

on $\mathcal{M}(K^{(\delta)})$, which will not lose the generality (see Appendix C). We show the following Lemma:

LEMMA 4.5. *Let $\mathcal{M}_k(K^{(\delta)})$ be the $2 \times k$ submatrix*

$$(4.21) \quad \mathcal{M}_k(K^{(\delta)}) = \begin{pmatrix} v_0^{(\delta)} - v_2^{(\delta)} & \cdots & v_{k-1}^{(\delta)} - v_{k+1}^{(\delta)} \\ v_1^{(\delta)} & \cdots & v_k^{(\delta)} \end{pmatrix} \quad (k = 1, \dots, \delta - 1)$$

of $\mathcal{M}(K^{(\delta)})$ subject to (4.20). If (4.19) with $m = k+1$ hold true under $\text{rank} \mathcal{M}_k(K^{(\delta)}) = 1$, then so does $\text{rank} \mathcal{M}_{k+1}(K^{(\delta)}) = 1$.

PROOF. We write down one of the assumption, (4.19) with $m = k+1$, more explicitly as

$$(4.22) \quad \sum_{j=0}^{\lfloor \frac{k+1}{2} \rfloor - 1} \left(\sum_{n=j+1}^{(k+1)-j-1} B_n^{(\delta, k+1)} \right) \Delta_{j+1, k+1-j}^{(\delta)} \\ = \left(\sum_{n=1}^{(k+1)+1} B_n^{(\delta, k+1)} \right) \Delta_{1, k+1}^{(\delta)} + \sum_{j=1}^{\lfloor \frac{k+1}{2} \rfloor - 1} \left(\sum_{n=j+1}^{(k+1)-j-1} B_n^{(\delta, k+1)} \right) \Delta_{j+1, k+1-j}^{(\delta)} = 0.$$

Since the other assumption, $\text{rank} \mathcal{M}_k(K^{(\delta)}) = 1$, implies

$$(4.23) \quad \Delta_{j+1, k+1-j} = 0 \quad (j = 1, \dots, \lfloor \frac{k+1}{2} \rfloor - 1),$$

we can bring (4.22) into

$$(4.24) \quad \left(\sum_{n=1}^{(k+1)+1} B_n^{(\delta, k+1)} \right) \Delta_{1, k+1}^{(\delta)} = 0.$$

so that we have $\Delta_{1, k+1} = 0$. The vanishment $\Delta_{1, k+1} = 0$ is put together with $\text{rank} \mathcal{M}_k(K^{(\delta)}) = 1$ to show $\text{rank} \mathcal{M}_{k+1}(K^{(\delta)}) = 1$. This completes the proof. \square

We are at the final stage to draw (1.3) from (4.9): Let us consider (4.19) with $m = 2$ for $K^{(\delta)}(q, p)$ subject to (4.20), which is written explicitly as $\Delta_{1, 2}^{(\delta)} = 0$. This shows $\text{rank} \mathcal{M}_2(K^{(\delta)}) = 1$ under (4.20). We can thereby start applying Lemma 4.5 to (4.19) recursively from $m = 3$ to $m = \delta - 1$, and finally reach to $\text{rank} \mathcal{M}(K^{(\delta)}) = \text{rank} \mathcal{M}_{(\delta-1)+1}(K^{(\delta)}) = 1$ under (4.20). As for $K^{(\delta)}(q, p)$ not subject to (4.20), Appendix C shows that (1.3) is a necessary condition of (4.9). Recalling Theorem 2.2, we have the following.

THEOREM 4.6. *If a δ -PHO shares its BGNF up to degree- $(2\delta - 2)$ with a $(2\delta - 2)$ -PHO up to degree- $(2\delta - 2)$, then the δ -PHO is separable within a rotation of Cartesian coordinates.*

Theorems 4.4 and 4.6 are put together to make our conclusion:

CONCLUSION. Our main theorem, Theorem 1.2, holds true.

Appendix A. Outline of the proof of Lemma 3.5

Since the BGNF of the 3-PHO Hamiltonians has been given explicitly¹⁰ in [U1], we focus our attention only to the case of $\delta \geq 4$ henceforth.

(i) Equating the homogeneous parts of degree-3 on the both sides of (3.4), we have $G_3^{(\delta)}(q, \eta) + (D_{q,\eta}^{(3)} W_3^{(\delta)})(q, \eta) = 0 (= H_3^{(\delta)}(q, \eta))$. On account of $\ker D_{q,\eta}^{(3)} = \{0\}$ ¹¹, the equation above for the degree-3 part is solved as $G_3^{(\delta)}(\xi, \eta) = 0$ and $W_3^{(\delta)}(q, \eta) = 0$. Then by induction, we can show

$$(A.1) \quad G_\ell^{(\delta)}(\xi, \eta) = 0 \quad W_\ell^{(\delta)}(q, \eta) = 0 \quad (\ell = 3, \dots, \delta - 1).$$

Under (A.1), the degree- δ part of (3.4) takes the form $G_\delta^{(\delta)}(q, \eta) + (D_{q,\eta}^{(\delta)} W_\delta^{(\delta)})(q, \eta) = V^{(\delta)}(q) (= H_\delta^{(\delta)}(q, p))$, so that we have

$$(A.2) \quad G_\delta^{(\delta)}(\xi, \eta) = V^{(\delta)\ker}(\xi, \eta) \quad W_\delta^{(\delta)}(q, \eta) = (\tilde{D}_{q,\eta}^{(\delta)-1} V^{(\delta)\text{image}})(q, \eta)$$

where $\tilde{D}_{q,\eta}^{(\delta)}$ denotes the restrict of $D_{q,\eta}^{(\delta)}$ on its image. This shows (3.9).

(ii) Under (A.1) and (A.2), we can show

$$(A.3) \quad G_\ell^{(\delta)}(\xi, \eta) = 0 \quad W_\ell^{(\delta)}(q, \eta) = 0 \quad (\ell = \delta + 1, \dots, 2\delta - 3)$$

by induction. On substituting (A.1)-(A.3) into (3.4), the degree- $(2\delta - 2)$ part of (3.4) is calculated to be

$$(A.4) \quad G_{2\delta-2}^{(\delta)}(q, \eta) + (D_{q,\eta}^{(\delta)} W_{2\delta-2}^{(\delta)})(q, \eta) = \frac{1}{2} \sum_{j=1}^2 \left\{ \left(\frac{\partial W_\delta^{(\delta)}}{\partial q_j} \right)^2 - \left(\frac{\partial W_\delta^{(\delta)}}{\partial \eta_j} \right)^2 \right\}.$$

The final expression (3.11) with (3.12) is obtained by writing down explicitly the kernel component of the rhs of (A.4), which requires another very simple but long calculation.

Appendix B. Proof of (4.15)

From (4.14), the rhs of (4.15) is put in a form

$$(B.1) \quad \begin{aligned} & \left((\delta - 1)^2 \binom{\delta-2}{m-1} \right)^{-1} \sum_{n=0}^m B_n^{(\delta,m)} \\ &= \sum_{k=0}^{\delta-2} \frac{\binom{\delta-2}{k}}{\{2(k+1) - \delta\}(2k - \delta)} \left(\sum_{n=N_1^b}^{N_1^\#} \binom{m-1}{n} \binom{\delta-m}{n+k+1-m} \right) \\ & \quad + \sum_{k=0}^{\delta-2} \frac{\binom{\delta-2}{k}}{\{2(k+2) - \delta\}\{2(k+1) - \delta\}} \left(\sum_{n=N_2^b}^{N_2^\#} \binom{m-1}{n-1} \binom{\delta-m}{n+k+1-m} \right) \\ & \quad + \sum_{k=0}^{\delta-1} \frac{\binom{\delta-1}{k}}{\{2(k+1) - \delta\}(2k - \delta)} \left(\sum_{n=N_3^b}^{N_3^\#} \binom{m}{n} \binom{\delta-m-1}{n+k-m} \right), \end{aligned}$$

¹⁰The expression agrees with (3.8)-(3.11).

¹¹There is no invariant homogeneous polynomials of odd-degree under the $SO(2)$ action generated by $D_{q,\eta}$. viz $\ker D_{q,\eta}^{(k)} = \{0\}$ for any odd k .

where

$$(B.2) \quad \begin{aligned} N_1^b &= \max(0, m-1-k) & N_1^\sharp &= \min(m-1, \delta-1-k) \\ N_2^b &= \max(1, m-1-k) & N_2^\sharp &= \min(m, \delta-1-k) \\ N_3^b &= \max(0, m-k) & N_3^\sharp &= \min(m, \delta-1-k). \end{aligned}$$

For the sums with the summation index n on the rhs of (B.1), we have

$$(B.3) \quad \begin{aligned} \sum_{n=N_1^b}^{N_1^\sharp} \binom{m-1}{n} \binom{\delta-m}{n+k+1-m} &= \sum_{n=N_3^b}^{N_3^\sharp} \binom{m}{n} \binom{\delta-m-1}{n+k-m} = \binom{\delta-1}{k} \\ \sum_{n=N_2^b}^{N_2^\sharp} \binom{m-1}{n-1} \binom{\delta-m}{n+k+1-m} &= \binom{\delta-1}{k+1}. \end{aligned}$$

Hence, (B.1)-(B.3) are put together to show

$$(B.4) \quad \begin{aligned} \frac{\sum_{n=0}^m B_n^{(\delta, m)}}{(\delta-1)^2 \binom{\delta-2}{m-1}} &= \left[\sum_{k=0}^{\delta-2} \frac{\binom{\delta-1}{k} \binom{\delta-2}{k}}{\{2(k+1)-\delta\} \{2k-\delta\}} \right. \\ &\quad + \sum_{k=0}^{\delta-2} \frac{\binom{\delta-1}{k+1} \binom{\delta-2}{k}}{\{2(k+2)-\delta\} \{2(k+1)-\delta\}} \\ &\quad \left. + \sum_{k=0}^{\delta-1} \frac{\binom{\delta-1}{k} \binom{\delta-1}{k}}{\{2(k+1)-\delta\} \{2k-\delta\}} \right] \\ &= \sum_{k=1}^{\delta-2} \frac{\binom{\delta-1}{k} \{ \binom{\delta-2}{k} + \binom{\delta-2}{k-1} - \binom{\delta-1}{k} \}}{\{2(k+1)-\delta\} \{2k-\delta\}} = 0. \end{aligned}$$

Appendix C. $SO(2)$ action to $\mathcal{M}(K^{(\delta)})$

To make the following discussion simple, we assume that δ is odd. Let $K'^{(\delta)}(q, p)$ be the δ -PHO Hamiltonian given by

$$(C.1) \quad K'^{(\delta)}(q, p) = K^{(\delta)}(\sigma(\psi)^{-1}q, \sigma(\psi)^{-1}p),$$

where $K^{(\delta)}(q, p)$ is defined by (1.2), and $\sigma(\psi)$ by (2.18). It is then shown by a straightforward calculation that they are related as

$$(C.2) \quad \mathcal{M}(K'^{(\delta)}) = \sigma(2\psi) \mathcal{M}(K^{(\delta)}) R^{(\delta-2)}(\psi),$$

where $R^{(\delta-2)}(\psi)$ is the standard representation of $SO(2)$ on the real vector space of homogeneous polynomials of degree $(\delta-2)$.

Assume that $K^{(\delta)}(q, p)$ is not subject to (4.20). What we have to show is the existence of $\sigma(\psi)$ which brings $K^{(\delta)}(q, p)$ to $K'^{(\delta)}(q, p)$ with $\mathcal{M}(K'^{(\delta)})$ having non-vanishing first column. Since the non-existence of such $\sigma(\psi)$ is equivalent to that the vector subspace, $\mathcal{N} = \text{span}\{q_1^h q_2^{\delta-2-h}\}_{h=1, \dots, \delta-2}$, is an invariant subspace of the $SO(2)$ action given by $R^{(\delta-2)}(\psi)$. However, this is not true: It is easily seen that any invariant subspace is given by a direct sum of the 2-dimensional subspaces each of which is spanned by $\Re((q_1 + iq_2)^h (q_1 - iq_2)^{\delta-2-h})$ and $\Im((q_1 + iq_2)^h (q_1 - iq_2)^{\delta-2-h})$ ($h = 0, \dots, (\delta-1)/2$). Hence \mathcal{N} is not $SO(2)$ -invariant.

We are now at the final stage to explain that the assumption (4.20) does not lose generality of our proof of the necessity. Let consider the δ -PHO Hamiltonian $K^{(\delta)}(q, p)$ which is not subject to (4.20) and shares its BGNF with a $(2\delta - 2)$ -PHO $K^{(2\delta-2)}(q, p)$. As shown above, we can bring $K^{(\delta)}(q, p)$ to $K'^{(\delta)}(q, p)$ through (C.1) that satisfies (4.20) by adopting a suitable rotation with $\sigma(\psi)$. According to the commutativity of the BG normalization and the rotations shown in Section 3, $K'^{(\delta)}(q, p)$ shares its BGNF with a $(2\delta - 2)$ -PHO $K'^{(2\delta-2)}(q, p)$ other than $K^{(2\delta-2)}(q, p)$. Hence, the discussion in subsection. 4.2 is applied to $K'^{(\delta)}(q, p)$ to show the necessity of $\text{rank}\mathcal{M}(K'^{(\delta)}) = 1$. Accordingly, the equation,

$$(C.3) \quad \text{rank}\mathcal{M}(K'^{(\delta)}) = \text{rank}\mathcal{M}(K^{(\delta)}),$$

following from (C.2) leads us to (1.3).

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